Physics of wakefields of very short bunches

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Outline of the talk

The main thesis of this talk: while calculating wakefields of very short bunches is a challenging computational problem, using approximations that take into account the smallness of σ_z can greatly facilitate the job and add additional insight into the physics of wakefields.

- Specificity of wakes for short bunches
- Optical model
- Parabolic equation (PE) for calculation of wakefields
- Scaling properties of the impedance in PE
- Combining computer simulations and analytic wakes

Motivation: short bunches

RMS bunch lengths in future lepton accelerators

PEP-X	5 mm
CEPC	3 mm
TLEP-W	2.2 mm
ILC	300 μm
LCLS-II	1000, 270, 25 μm

Calculation of wakefields is more difficult for long, small-angle tapers.

The difficulty is associated with a small parameter σ_z/b, where b is the typical size of the structure that generates the impedance (say, iris radius in RF cavity). On the other hand the small parameter allows us to develop approximate analytical theories and use them for numerical calculations.

Catch-up distance is important for short bunches
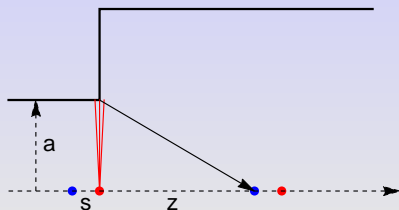
• If head particle passes e.g. the beginning of a cavity, tail particle doesn't know it until $z = l_{c-u} \sim a^2/2s$ (a beam pipe radius, s separation of particles) later. If $a = 3.5$ cm and $s = 25$ μm, then $z \approx 25$ m.

Hence, the steady state wake develops over the distance l_{c-u}, which can also be called the *formation length* of the wake.

• Transient region: there will be a transient regime before steady-state is reached; for Gaussian with length σ_z, transient will last until $z \sim a^2/2\sigma_z$.

• Wake is typically taken to act instantaneously. If the catch-up distance is not small compared to the betatron wavelength, the usual approach to collective beam dynamics should be modified.

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Optical approximation

The wake in bunch of length σ_z is formed by wavelengths $k \sim 1/\sigma_z$.
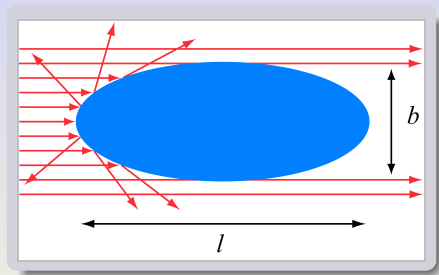
In electromagnetic theory the limit $k \to \infty$ corresponds to geometrical optics (the wavelength is much smaller than the size of the objects). Hence in the limit $\sigma_z \to 0$ there should be an analog of optical theory for wakefields.

A general theory of wakefields in optical approximation was developed in[1]. The advantage of this approach is that it allows to easily calculate the wakes for even 3D, non-axisymmetric geometries. This method works well if there are protrusions or sharp transitions in the vacuum chamber.

[1] Stupakov, Bane, Zagorodnov, PRST-AB **10**, 054401 (2007); Bane, Stupakov, Zagorodnov, PRST-AB **10**, 074401 (2007).

Impedance and wake in optical approximation

In the optical regime:

$Z_\parallel$ is real and independent of frequency; wake of a point charge $w_\parallel \propto \delta(z)$ and wake of a bunch with distribution $\lambda(z)$:

$$W_\parallel(z) \propto \lambda(z)$$

$Z_\perp$ is also real and depends on frequency as ω^{-1}; point charge wake $w_\perp \propto h(z)$, and wake of bunch distribution is

$$W_\perp(z) \propto \int^z \lambda(z')\,dz'$$
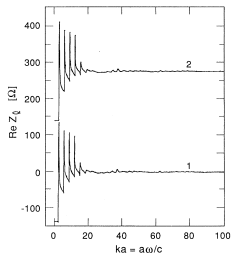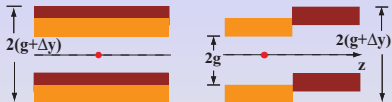
(Figure from [2]).

FIG. 10. The real part of the longitudinal coupling impedance of a cross section step as a function of parameter $ka = a\omega/c$; $b/a = 0.1$; the matrix size is 60×60; (1) $\mathrm{Re}Z_{in}$, (2) $\mathrm{Re}Z_{out}$.

The longitudinal impedance of a step transition does not depend on ω at high frequencies.
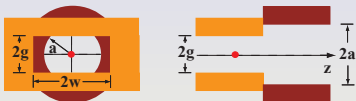
[2] Heifets, Kheifets, RMP, **63**, 631, 1991.

More Complicated Transitions
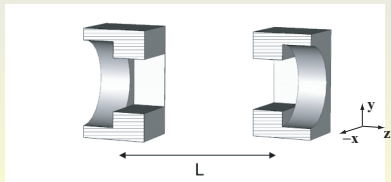

X1: misaligned flat beam pipes

X2: LCLS type rectangular-to-round transition

Cases considered:

- misaligned flat beam pipes
- LCLS rectangular-to-round transition

Cross-section view (left) and longitudinal view (right) of rectangular-to-round transition.

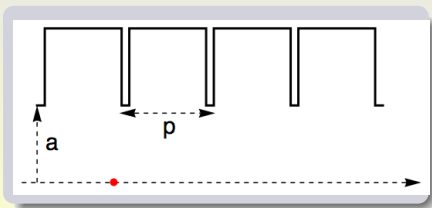A pair of LCLS transitions in perspective view.

Limitations of the optical model

The optical theory ignores diffraction effects. It predicts zero impedance for the pillbox cavity or periodic irises; the wake in these cases in the limit $\omega \to \infty$ is due to diffraction.
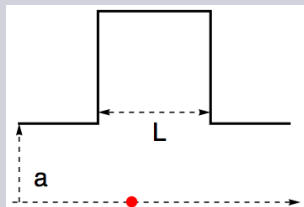
Pillbox cavity. Diffraction theory gives

$$Z_\parallel(k) = \frac{Z_0(1+i)}{2\pi^{3/2}}\sqrt{\frac{L}{ka^2}}$$

Periodic structure with thin irises ($Z_\parallel$ per unit length)

$$Z_\parallel(k) = \frac{iZ_0}{\pi ka^2}$$
$$\times \left(1 + 0.46(1+i)\sqrt{\frac{\pi p}{ka^2}}\right)^{-1/2}$$

Limiting value of wake for very short bunches

• Because the limit of high frequencies corresponds to small distances, we can infer the wake of a point charge at short distance behind it. For infinitely long cylindrically symmetric disk-loaded accelerator structure, the steady-state wakes at the origin is

$$w_{\parallel}(s) \approx \frac{Z_0 c}{\pi a^2}, \qquad w_{\perp}(s) = \frac{2Z_0 c}{\pi a^4} s, \qquad s \ll s_0$$

• This is also true for a resistive pipe (a is the pipe radius), a pipe with small periodic corrugations, and a dielectric tube within a pipe; it appears to be a general property [3].

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[3] S.S. Baturin and A.D. Kanareykin, arXiv:1308.6228 [physics.acc-ph] (2014).

Outline of the talk

- Specificity of wakes for short bunches
- Optical model
- Parabolic equation and scaling properties
- Combining computer simulations and analytic wakes

Parabolic equation

The parabolic equation is used:

- In diffraction theory. Proposed by M. A. Leontovich in 1944. Applied to various diffraction problems by V. Fock in 40-50.
- In the FEL theory.
- To compute synchrotron radiation of relativistic particles in toroidal pipe[4].

Synchrotron radiation of relativistic particles can be treated using the parabolic equation [5].

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[4] Stupakov, Kotelnikov, PRST-AB **6**, 034401 (2003); Agoh, Yokoya, PRST-AB **7**, 054403 (2004).

[5] Geloni et al., DESY Report 05-032, (2005).

Parabolic equation

The Fourier transformed electric field $\hat{\mathbf{E}}$ and the longitudinal component of the current $\hat{j}_s$ are written with the additional factor e^{-iks}:

$$\hat{\mathbf{E}}(x, y, s, \omega) = e^{-iks} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \mathbf{E}(x, y, s, t)$$

$$\hat{j}_s(x, y, s, \omega) = e^{-iks} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, j_s(x, y, s, t)$$

where $k \equiv \omega/c$. One also introduces the transverse component of the electric field $\hat{\mathbf{E}}_\perp$ as a two-dimensional vector $\hat{\mathbf{E}}_\perp = (\hat{E}_x, \hat{E}_y)$, and the longitudinal component of the electric field $\hat{E}_s$.

It is assumed that $\hat{\mathbf{E}}_\perp$ $\hat{j}_s$ are "slow" functions of s, such that $\partial/\partial s \ll k$. It means that we are interested in components of the field propagating in the positive direction of s at small angles to the axis. In particular, we neglect a part of the field propagating in the negative direction of s.

Parabolic equation

From the wave equation for the field it follows that [6]

$$\frac{\partial}{\partial s}\hat{\mathbf{E}}_\perp = \frac{i}{2k}\left(\nabla_\perp^2\hat{\mathbf{E}}_\perp + \frac{2k^2x}{R}\hat{\mathbf{E}}_\perp - \frac{4\pi}{c}\nabla_\perp\hat{j}_s\right)$$

where $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$, R is the radius of curvature (for a straight pipe $R^{-1} \to 0$, $s \to z$). The longitudinal electric field can be expressed through the transverse one and the current

$$\hat{\mathbf{E}}_s = \frac{i}{k}\left(\nabla_\perp \cdot \hat{\mathbf{E}}_\perp - \frac{4\pi}{c}\hat{j}_s\right)$$

A remarkable feature of this equation is that $\hat{\mathbf{E}}_\perp$ varies in s over the distance much larger than $\lambda = k^{-1}$.

In contrast to the optical approximation PE takes into account diffraction effects (the pillbox impedance is derivable from PE). It is valid for high frequencies, and especially good for small-angle transitions.

[6]G. Stupakov, New Journal of Physics **8**, 280 (2006); G. Stupakov, Reviews of Accelerator Science and Technology **3**, 3956 (2010).

Impedance scaling in PE

Analysis shows that the longitudinal impedance $Z_L(\omega)$ in a small-angle geometry (3D, in general), with characteristic length L in z-direction is

$$Z_L(\omega) = F\left(\frac{\omega}{L}\right)$$

Compute impedance for a short structure, $Z_{\frac{1}{n}L}$, and use the scaling law

$$Z_L(\omega) = Z_{L/n}\left(\frac{\omega}{n}\right)$$

Translating the impedance into the longitudinal wake we find

$$w_{L,\sigma_z}(s) = n w_{L/n, n\sigma_z}(ns)$$

For the transverse wake

$$\boldsymbol{w}_{L,\sigma_z}^{(t)}(s) = \boldsymbol{w}_{L/n, n\sigma_z}^{(t)}(ns)$$

The computational time in 2D reduces by n^3.

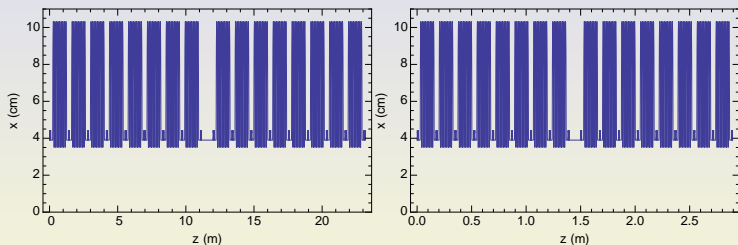## Practical example of using the scaling property

The nominal LCLS-II bunch length is $\sigma_z = 25$ μm. The beam is accelerated in SC RF cavities, with a cryomodule housing 8 nine-cell cavities. The length of the cryomodule is ~12 m. It is important to calculate the cavity heating due to the energy deposited by the beam through the wakefield.
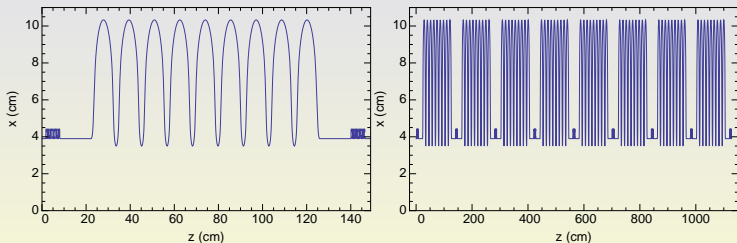
Practical example of using the scaling property

One wake was calculated with $\sigma_z = 25$ μm for two cryomodules (3.5 days run time), the other was calculated for $\sigma_z = 200$ μm in the cryomodule geometry shrunk 8 times longitudinally (40 min run time).
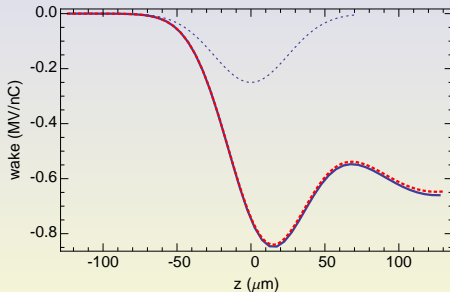

Real geometry (left) and scaled geometry (right).

Practical example of using the scaling property

Surprisingly, the scaling works very well for the cavities.

$$w_L(s) = 8 w_{\frac{1}{8}L}(8s)$$

After rescaling the results agree very well!

Outline of the talk

## Combining computer simulations with analytics

Recently[7] a method was suggested to calculate short bunch wake-potentials, and even point-charge wakefields, running an EM solver for a relatively long bunch. This approach can save greatly on calculation speed and provides physics insights.

The idea behind the method is to use a combination of computer simulations with an analytical form of the wakefield for a given geometry in the limit $\sigma_z \to 0$.

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[7] Podobedov, Stupakov, PRST-AB **16**, 024401 (2013)

Basic idea illustrated on step-out transition

Consider a particular example of the wake-potential of a short bunch passing through a step-out transition from radius r_{min} to r_{max}.
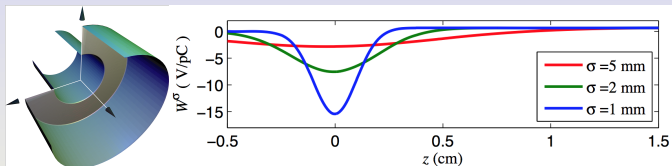
The plot of the wake-potential $W^\sigma(z)$ in this case, for several values of σ, is shown. With decreasing σ, the wake-potential becomes larger inside the bunch; in the limit $\sigma \to 0$, it diverges as $1/\sigma$. The singular part of the wake in the limit $\sigma \to 0$ is provided by the optical model.

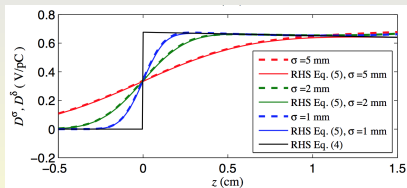# Basic idea

In the limit $\sigma \to 0$ the optical model gives

$$W_s^\delta(z) = -\frac{Z_0 c}{\pi} \delta(z) \ln \frac{r_{max}}{r_{min}}, \qquad W_s^\sigma = -\frac{Z_0 c}{2^{1/2} \pi^{3/2} \sigma} \ln \frac{r_{max}}{r_{min}} e^{-z^2/2\sigma^2}$$

Subtracting it from the wake we introduce the difference

$$D^\sigma(z) = W^\sigma(z) - W_s^\sigma$$

Plot of $D^\sigma(z)$

When $\sigma \to 0$ this function approaches a well defined limit shown by the solid line. We denote this limit by $D^\delta(z)$,
$D^\delta(z) = \lim_{\sigma \to 0} D^\sigma(z)$.

Basic idea

In the vicinity of point $z = 0$ D^δ can be approximated

$$D^\delta(z) \approx (\alpha + \beta z)h(z),$$

where $H(z)$ is the step function ($h = 1$ for $z > 0$ and $h = 0$ otherwise). Then

$$\begin{aligned}
D^\sigma(z) &= \int dz' \lambda(z + z') D^\delta(z') \\
&= \frac{\alpha + \beta z}{2}\left(1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}\sigma}\right)\right) + \frac{\beta\sigma}{\sqrt{2\pi}}e^{-z^2/2\sigma^2}
\end{aligned}$$

The crucial element of the method is that α and β can be obtained from simulations running a relatively long bunch through the system and fitting to the formula above.

Basic idea

Comparing this with the simulated $D^\sigma(z)$ in the region $z < 3\sigma$ one can find the parameters α and β and thus to establish the dependence of $D^\delta(z)$ in this region. After $D^\delta(z)$ is found, we have the wakefield of the point charge

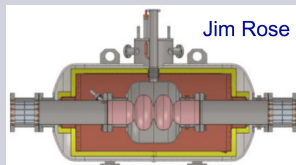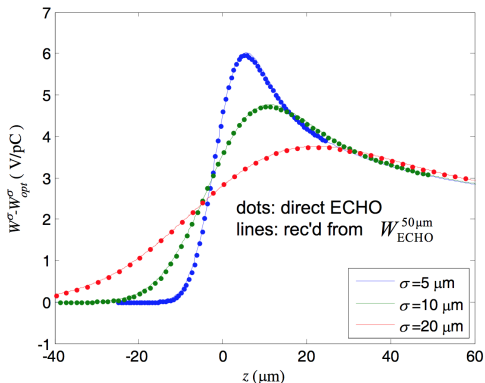$$W^\delta(z) = W_s^\delta + D^\delta(z),$$

[note that W_s^δ is a delta-function].

The particular form of the singular part of the wake-potential, W_s^δ, is determined by the high-frequency limit of the impedance for a given geometry; in most cases it can be found in the literature.

Practical example: NSLS-II Landau cavity

- 1.5 GHz dual cell cavity, $r_{side\ pipe} = 6$ cm
- Final results for the short-range wakes:

Jim Rose

To find 10 μm bunch wake: Brute force: 480 hours of Intel(R) Xeon(R) 5570@2.93 Ghz CPU to $z_{max} = 1$ cm. Our method: uses only $\sigma = 50$ μm bunch, saves a factor of 5^3 on CPU time and 5^2 on memory. Gives a model of the point-charge wake as a bonus.

Conclusions

- For large and smooth accelerator structures, and short bunches, direct EM solver calculations can be extremely time and memory-consuming. Using approximate methods that employ small geometric parameters in the problem greatly facilitates the numerical solution.

- Optical approximation and parabolic equation are the new approaches that try to address the issue of wakefield for very short bunches.

- A new method that combines a (processed) long-bunch wake from an EM solver and a singular analytical wake model allows one to accurately obtain wakefields of short bunches, including that of a point-charge.

Discussion

High-repetition superconducting linacs add a new dimension to the problem of wakefields. The EM energy released by electron bunches is eventually deposited somewhere inside the vacuum chamber. The goal is to reliably calculate to where this energy goes.

Two issues:

- Tracing propagation of high-frequency EM waves inside the linac is a very difficult problem.
- A large fraction of this energy is at relatively low frequencies.

300 pC, 1MHz, integrated power
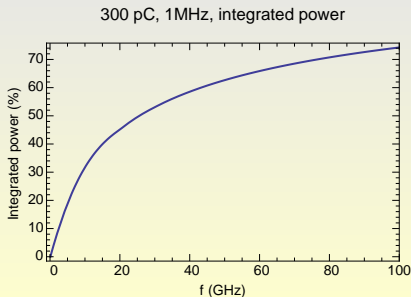
We think that the most promising approach is based on using S-matrix formalism and working in frequency domain (K. Bane's talk this morning).

Estimate of power from the LCLS-II BC2

Steady state CSR model:

$$P_{CSR} = \varkappa_{CSR} L Q^2 f_{rep}$$

with

$$\varkappa_{CSR} = 0.76 \frac{Z_0 c}{2 \cdot 3^4/3\pi} \frac{1}{\rho^{2/3} \sigma_z^{4/3}}$$

The last magnet of BC2. Shielding is not important. Estimated radiation power: $Q = 300$ pC, repetition rate $f_{rep} = 1$ MHz.

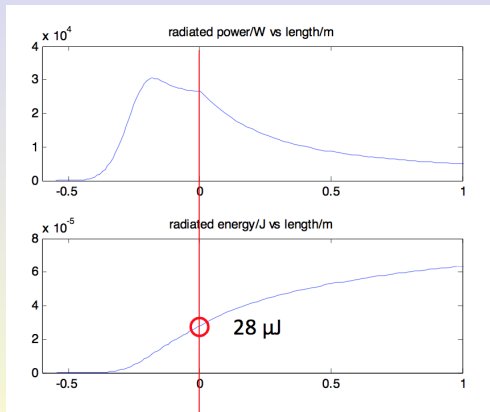| E [GeV]	1.6
L [m]	0.55
ρ [m]	10.2
σ_z [μm]	24
P_{CSR} [W]	48.5

The steady-state model is valid if $L \gg \ell$:

$$\ell = (24 \sigma_z \rho^2)^{1/3} \approx 40 \text{ cm}$$

M. Dohlus: the beam keeps loosing energy after exiting the bend. Calculation for CSR in free space:

Contribution from space after the magnet
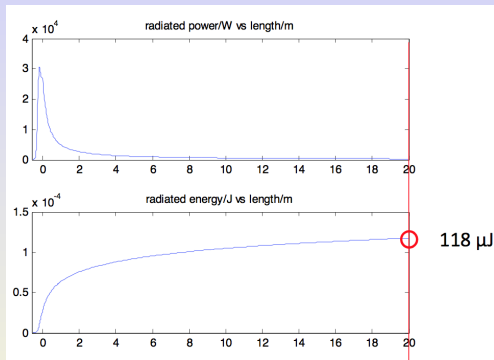
These are calculations without shielding. If one assumes $v = c$, the radiated energy slowly grows to ∞ as $z \to \infty$. Accounting for finite (but large γ) would limit the growth by $\sim \ln \gamma$. However the shielding would also become important here, $\sim \ln r_{\mathrm{pipe}} / \sigma_z$.

I have a code which computes the CSR wakefield in a bend in a rectangular vacuum chamber (PRST-AB,2009).
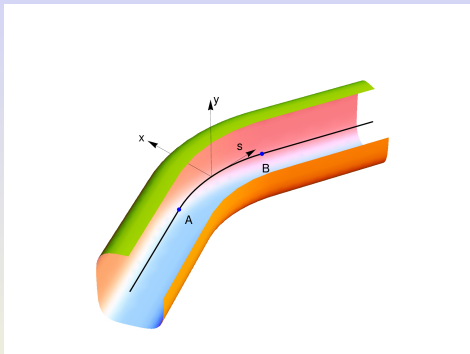
A vacuum chamber was assumed with full vertical gap $2h = 3.2$ cm, and the full horizontal gap 9.6 cm.
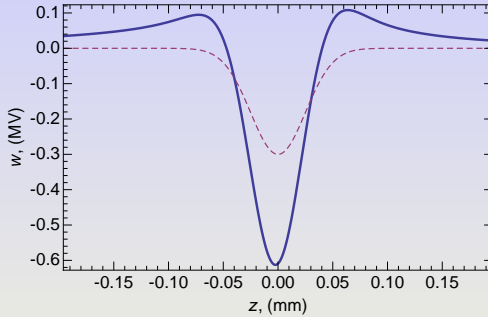
The beam radiates 113 μJ energy (113 W).

Discussion

- About 120 W will be radiated from the last bend of BC2 ($Q = 300$ pC, $f_{rep} = 1$ MHz).
- The linac L3 is about 60 m downstream BC2 with the cross section of the vacuum chamber changing from $\sim$ rectangular to round. What happens with this radiation as it propagates downstream? How much of it enters the SC linac? Can we (partially?) shield the linac from the radiation?
- Brute force simulation of propagation of this radiation downstream of BC2 may be prohibitively slow. Analysis that utilizes some kind of short-wavelength approximation (optical or diffraction models) might be more appropriate.